Langlands functoriality conj.

- Langlands-Shahidi
- Trace formula

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I. Intro. (LLC / L-ftns)

II. Linear algebraic groups (GL_n)

III. Unramified LLC

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VII Langlands Functoriality conj.

VIII Converse theorem

& Arthur's trace formula

L 1

Intro: LLC with L-fns

L1. Introduction (Preliminaries)

a. Rep'n theory

Def 1.1 (π, V) : a rep'n of H

v.s./$\mathbb{C}$

a gp homomorphism $\pi: H \to \mathrm{Aut}_{\mathbb{C}}(V)$

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Def 1.2 An irred rep'n (π, V) of H:

π
no proper $W \subset V$ (no H-stable subspace W $0 \neq W \neq V$)

Q What kinds of rep'ns in Langlands program?

L1. Introduction (Preliminaries)

a. Rep'n theory

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V v.s./$\mathbb{C}$
a gp homomorphism $\pi: H \to \mathrm{Aut}_{\mathbb{C}}(V)$

Def 1.2 An irred rep'n (π, V) of H:

no proper $W \subset V$ (π acting on V) (no H-stable subspace W, $0 \neq W \neq V$)

Q What kinds of rep'ns in Langlands program?

A
- (local) admissible rep'ns of GL_n
  (Especially) generic rep'n (Langlands-Shahidi) (Classical gps, GSpin gps recent result)
- Galois rep'ns
- (global) Automorphic rep'ns

b. Local Langlands corr. (without def)

$$\left\{\begin{array}{l}\text{admissible} \\ \text{rep'n of } G\end{array}\right\} \xrightarrow{\text{finite to one map}} \left\{\begin{array}{l}\text{Langlands parameter} \\ \text{(Galois rep'n)}\end{array}\right\}$$

$\uparrow$ conn. red gp $/F$ $\leftarrow$ char 0.

Analytic side
(Harmonic analysis)
(representation theory)

Number theory side
Galois gp &
Galois representation.

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$\uparrow$ conn. red gp $/F$ $\leftarrow$ char 0.

Analytic side
(Harmonic analysis)
(representation theory)

Number theory side
Galois gp &
Galois representation.

L-ftns (if we can define)? $=$ Artin L-ftns

- Langlands-Shahidi ⁰ via Fourier coefficient of Eisenstein series
- Rankin-Selberg method ⁰ via integral representation.

Projects

c. L-functions

"Dirichlet series"

$a_1, a_2, a_3, \cdots \rightsquigarrow L(s, X) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$

Data of X

"cpx numbers"

ex) variety, elliptic curve, modular forms, automorphic forms

Q

Ex 1.3 Riemann-zeta ftn ($a_n = 1 \ \forall n$)

$\zeta(s) :=$

(P1) Euler Product

(P2) Functional Equation:

(P3) Meromorphic continuation to $\mathbb{C}$ with a pole at $s = 1$.

Def

Ex 1.4 E : elliptic curve / $\mathbb{Q}$

$$y^2 = x^3 + ax + b \quad \left(a, b \in \mathbb{Q},\ -16(4a^3 + 27b^2) \neq 0 \right)$$

$-16(4a^3 + 27b^2) \to \Delta$

$$a_p(E) = \begin{cases} p+1 - \#E(\mathbb{F}_p) & \text{if } p \nmid \Delta \\ -1, 0, 1 & \text{if } p \mid \Delta \end{cases}$$

Define $L_p(s, E) := (1 - a_p(E) p^{-s} + \delta(p) p^{1-2s})^{-1}$

$$\delta(p) = \begin{cases} 1 & p \nmid \Delta \\ 0 & p \mid \Delta \end{cases}$$

Define $L(s, E) := \prod_p L_p(s, E)$

i.e.

Q. (P2) and (P3) for $L(s, E)$?

Hasse – Weil conjecture for E over $\mathbb{Q}$

A.

Ex 1.5 f: (cusp) modular form wt $2k$ w.r.t. $\Gamma_0(N)$

where $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$

modular property $\Rightarrow f(\gamma \cdot z) = (cz+d)^{2k} f(z)$

$\frac{az+b}{cz+d}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

With $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow f(z+1) = f(z)$

Fourier analysis $\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n(f) q^n$, $q = e^{2\pi i z}$

Define $L(s, f) = \sum_{n=1}^{\infty} a_n(f) n^{-s}$

(P1)

Fact: P2 & (P3)

Conj 1.6 Taniyama-Shimura-Weil conj

E: elliptic curve/$\mathbb{Q}$

$\exists$ modular form f s.t. $a_p(E) = a_p(f)$ $\forall p$.

$\Rightarrow L_p(s, E) = L_p(s, f)$

$$L(s, E) = L(s, f)$$

(P1) ✓ ✓

(P2) ---- ✓

(P3) - - - ✓

Q.

Q. Can we find other natural source of "nice" L-ftns?

(cuspidal) automorphic rep'ns of $GL(n)$

$n=1$: Hecke characters

Tate's thesis (Bump Sec 3.1)
through so-called zeta integral

$n=2$: ex) modular forms

$\vdots$

(through local zeta integral)

Ex 1.7 $L(s,\chi)$, $\chi: F^\times \to \mathbb{C}^\times$

F: non-archimedean local field

$\mathcal{O}$: the ring of integers

ϖ: a uniformizer of F

q: cardinality of residue field of F

$\mathfrak{p}$: maximal ideal of $\mathcal{O}$, i.e. $\varpi\mathcal{O} = \mathfrak{p}$

$k = \mathcal{O}/\mathfrak{p}$: residue field

Say χ is unramified if χ is trivial on $\mathcal{O}^\times$

(is of the form $\chi(a) = |a|_F^s$ for some $s \in \mathbb{C}$)

"p-adic valuation"

$$L(s,\chi) := \begin{cases} (1-\chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified} \\ 1 & \text{otherwise} \end{cases}$$

Q. Is it nice?

Q. Is it nice? Yes (Tate's thesis)

(See also Ex 1.10 and Remark 1.11 for global case)

How? very briefly,

$S(F)$: the space of Schwartz-Bruhat fns on F

$= \{$ locally constant, cptly supported if F non-archimedean
rapidly decreasing fns if F archimedean

$Z(s, \phi, \chi) :=$

$\{ Z(s, \phi, \chi) : \phi \in S(F) \}$: family of local zeta integrals

$L(s, \chi)$: GCD of this family, i.e.

① $\exists \phi \in S(F)$ s.t. $Z(s, \phi, \chi) = L(s, \chi)$ for χ unramified

② $Z(s, \phi, \chi)/L(s, \chi)$ is entire $\forall \phi \in S(F)$.

Note If χ is ramified, then $Z(s, \phi, \chi)$ is entire.

연습

Ex 1.8 $\phi = \phi_0 =$ characteristic f-tn of $\mathcal{O}$

$$Z(s, \phi_0, \chi)$$

$$= \begin{cases} \dfrac{1}{1-\chi(\varpi)q^{-s}} & \text{if } \chi \text{ is unramified} \\ 0 & \text{if } \chi \text{ is ramified.} \end{cases}$$

Remark 1.9 (local zeta integral)

① $Z(s, \phi, \chi)$ convergent if $\mathrm{Re}(s) > 0$

② $Z(s, \phi, \chi)$ has meromorphic continuation to all $s \in \mathbb{C}$.

③ $Z(s, \phi, \chi)$ satisfies local functional equation.

i.e.

Ex 1.10

restricted product relative to $\mathcal{O}_v$ / $\mathcal{O}_v^\times$

K : number field, $\mathbb{A} := \prod_v{}' K_v$ the Adele

$\mathbb{A}^\times := \prod_v{}' K_v^\times$ the idele

$\underline{\chi}$: Hecke character (cont. character of $\mathbb{A}^\times / K^\times$)

$\underline{\chi} = \otimes \underline{\chi}_v$ where $\underline{\chi}_v = \underline{\chi} \circ i_v$

$i_v : K_v^\times \hookrightarrow K^\times \backslash \mathbb{A}^\times$

$\gamma_v \longmapsto (\cdots, 1, 1, \gamma_v, 1, 1, \cdots)$

$\underline{\chi}_v$ is unramified for almost all v.

Let S = finite set of places s.t.

① contains archimedean places

② $\underline{\chi}_v$ is unramified if $v \notin S$.

Define (partial) L-ftn associated to $\underline{\chi}$

$$L_S(s, \underline{\chi}) = \prod_{v \notin S} L(s, \underline{\chi}_v) = \prod_{v \notin S} (1 - \underline{\chi}_v(\overline{w}_v) q_v^{-s})^{-1}.$$

Remark 1.11 (Global zeta integral)

Let K, $\mathbb{A}$ be as in Ex 1.10.

$S(\mathbb{A})$: the space of Schwartz-Bruhat fns on $\mathbb{A}$.

Ψ

$\Phi = \bigotimes_v \Phi_v$ where

① $\Phi_v \in S(K_v)$

② $\Phi_v = \Phi_v^\circ$: the characteristic function of $\mathcal{O}_v$ for almost all v.

For $\Phi \in S(\mathbb{A})$,

$$Z(s, \Phi, \chi) = \int_{\mathbb{A}^\times} \Phi(x)\chi(x)|x|^s d^\times x$$

We formally have

$$Z(s, \Phi, \chi) = \prod_v Z(s, \Phi_v, \chi_v)$$

Ex 1.8 implies

$$Z(s,\underline{\chi},\underline{\Phi}) = \prod_{v\in S} Z(s,\underline{\chi}_v,\underline{\Phi}_v) \prod_{v\notin S}$$

$$= \prod_{v\in S} Z(s,\underline{\chi}_v,\underline{\Phi}_v) \prod_{v\notin S}$$

$$= \prod_{v\in S} Z(s,\underline{\chi}_v,\underline{\Phi}_v)\, L_S(s,\underline{\chi}). \quad (*)$$

Properties of Global zeta integral

(P1) $Z(s,\underline{\Phi},\underline{\chi})$ has meromorphic continuation to $\mathbb{C}$

(P2) The global functional equation:

$$Z(s,\underline{\Phi},\underline{\chi}) = Z(1-s,\hat{\underline{\Phi}},\underline{\chi}^{-1})$$

where $\psi : \mathbb{A}_K \to S^1$ a non-trivial additive character,

$$\hat{\underline{\Phi}}(y) = \int_{\mathbb{A}} \underline{\Phi}(x)\,\psi(xy)\,dx.$$

Let $L(s, \underline{\chi}) = \prod_v L(s, \underline{\chi}_v)$.

(P1) & (P2) (properties of global zeta integral)

implies meromorphic continuation of $L(s, \underline{\chi})$ and the following functional equation of $L(s, \underline{\chi})$

$$L(s, \underline{\chi}) = \varepsilon(s, \underline{\chi}, \psi) L(1-s, \underline{\chi}^{-1})$$

where $\varepsilon(s, \underline{\chi}, \psi) = \prod \varepsilon(s, \underline{\chi}_v, \psi_v)$.

pf of F.E. of $L(s, \underline{\chi})$

From (P2) and (*) in the previous page,

L2

Linear algebraic groups

Example 위주로 개념을 기억해두면 편함

L2. Linear algebraic groups (GLn case)

F : non-archimedean local field of char 0

Def 2.1 G: connected reductive gp/F

a Zariski-closed subgroup of some GL_N whose radical consists of only semisimple

"the largest connected solvable normal subgp of G"

(diagonalizable) elements.

Ex 2.2 GLn, classical gps (SO, Sp, U)

GSp ← radical : scalar matrix

Def 2.3 Borel subgroup of G is the the maximal connected solvable subgp.

Ex 2.4 G = GLn B =

Def 2.5 P: Parabolic subgp of G
closed sbgp of G that contains some Borel subgp.

$P = MN$ Levi decomposition

M: Levi subgroup (reductive gp)

N: unipotent radical of P

Ex 2.6 $G = GL_n$

P

B

Assume G is split.

i.e. G has a maximal torus T s.t.

T is isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$ over F ($\mathbb{G}_m \cong GL_1$)

Fix a Borel pair (B, T) (B: Borel containing T.)

$\Rightarrow B = TU$ (U: unipotent radical of B.)

determines $\rightsquigarrow$ the based root datum

$X^* := \mathrm{Hom}(T, \mathbb{G}_m)$ gp of chars of T

Δ^*: the set of simple roots of T in $\mathrm{Lie}(U)$

$X_* := \mathrm{Hom}(\mathbb{G}_m, T)$ gp of co-chars of T

Δ_*: the set of simple co-roots of T.

$\mathfrak{g} := \mathrm{Lie}(G)$: the Lie algebra of G

(ex. $\mathfrak{g} = n \times n$ matrix for $G = GL_n$)

$T \curvearrowright \mathfrak{g}$ adjoint action has the following decomposition.

$\mathrm{adj}(t)\, v = t v t^{-1}$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\chi} \mathfrak{g}_\chi \right)$$

$\mathfrak{g}_0$: T acts trivially

$\mathfrak{g}_\chi$: (kinds of) eigenspace with eigenvalue χ

$$\mathrm{adj}(t)\, v = \chi(t) v \quad \text{for} \quad \begin{matrix} t \in T \\ v \in \mathfrak{g}_\chi \end{matrix}$$

Such character χ of T is called roots.

Ex 2.7 $G = GL_2 / F$ ← non-archimedean field of char 0

maximal torus $T := \left\{ \begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix} \mid x_1, x_2 \neq 0 \right\}$

Then $X^* = \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_2$, where

$$(a\chi_1 + b\chi_2)\begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix} = x_1^a x_2^b$$

adjoint action ?

$\mathrm{ad}_{\begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} =$

$\alpha =$ _____ : the root $\rightsquigarrow$ $g_\alpha =$ _____

(resp. $-\alpha =$ _____) (resp. $g_{-\alpha} =$ _____)

$R = \{ \alpha = x_1 - x_2, \ -\alpha = x_2 - x_1 \}$

"the set of roots"

Def 2.8 (Positive roots w.r.t. (B, T)) $B = TU$

positive root $\sim$ $\mathrm{Lie}(U)$

R^+ : the set of positive roots.

Ex 2.9 $G = GL_2$

$$B = TU = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow \mathrm{Lie}(U) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$

Therefore R^+ w.r.t. (B, T) are

Def 2.10 (Simple roots)

Δ^* : the set of simple roots is the basis of R^+

연습

Ex 2.11 $G = GL_n$,

$$B = \begin{pmatrix} * & * & * & \cdots & * \\ & * & & & \vdots \\ & & \ddots & & \\ 0 & & & \ddots & \\ & & & & * \end{pmatrix} = \begin{pmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix} \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & & \vdots \\ & & \ddots & * \\ 0 & & & 1 \end{pmatrix}$$

$$\mathrm{Adj}\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \cdot \begin{pmatrix} 0 & & a^{(i,j)} \\ & \ddots & \\ 0 & & 0 \end{pmatrix} =$$

$R =$

Since $\mathrm{Lie}(U) = \begin{pmatrix} 0 & * & 0 & \cdots \\ & 0 & & \\ & & \ddots & \\ & & & \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & * & 0 & \cdots & 0 \\ & 0 & & & & \\ & & \ddots & & & \\ & & & & & \end{pmatrix} \oplus \cdots$,

$R^+ =$

$\Delta^* =$

Thm 2.12 (G, B, T) is classified by the based root datum $\Psi(G) = (X^*, \Delta^*, X_*, \Delta_*)$.

$(X^*, \Delta^*, X_*, \Delta_*) \Rightarrow (X_*, \Delta_*, X^*, \Delta^*)$

change the role

$\Updownarrow$

complex reductive gp $\hat{G}$ s.t. $(\hat{G}, \hat{B}, \hat{T}) \sim (X_*, \Delta_*, X^*, \Delta^*)$

Def 2.13 Such complex gp $\hat{G}$ is called the Langlands dual gp associated to G

	G	$\hat{G}$
A_n type	GL_n	$GL_n(\mathbb{C})$
B_n "	SO_{2n+1}	$Sp_{2n}(\mathbb{C})$
C_n "	Sp_{2n}	$SO_{2n+1}(\mathbb{C})$
D_n "	SO_{2n}	$SO_{2n}(\mathbb{C})$
B_n "	$GSpin_{2n+1}$	$GSp_{2n}(\mathbb{C})$
D_n "	$GSpin_{2n}$	$GSO_{2n}(\mathbb{C})$

L3

Unramified LLC

+ Basic representation theory

F: non-archimedean local field
$\mathcal{O}$: the ring of integer
$\mathfrak{p}$: the maximal ideal of $\mathcal{O}$ $\quad$ ϖ: uniformizer of F i.e. $\varpi\mathcal{O} = \mathfrak{p}$
$k = \mathcal{O}/\mathfrak{p}$: the residue field

Def 3.1 Locally profinite group

a topological gp G s.t. every open nbhd of the identity in G contains a cpt open sbgp of G.

$\Leftrightarrow$ locally cpt & totally disconnected

Ex 3.2 ① F non-archimedean local field

$\forall x \in F^\times \rightsquigarrow x = u\varpi^n$, $u \in \mathcal{O}^\times$ (unit), $n \in \mathbb{Z}$

$$\|x\| = q^{-n} = q^{-v_F(x)}$$

$\rightsquigarrow$ topology

$\mathfrak{p}^n = \varpi^n\mathcal{O} = \{x \in F \mid \|x\| \le q^{-n}\}$: open & cpt

give a fundamental system of open nbhd of 0 in F.

Fundamental system of open nbhd of 1 in .. (cpt)

(2) $F^\times$: $1+p^j\mathcal{O}$, $j\geq 1$

(3) $GL_n(F)$: $GL_n(\mathcal{O})$ & $1+p^jM_n(\mathcal{O})$, $j\geq 1$

(open subset of $M_n(F)$)

Def 3.3 smooth

(π, V) of G is smooth if

$\forall v \in V$, $\exists$ cpt open K of G s.t.

$v \in V^K$

↖ K-fixed vector

$\left(\Leftrightarrow V = \bigcup_K V^K \right)$

↑ cpt open sbgps of G

Def 3.4 admissible

A smooth (π, V) is admissible if

V^K is finite-dimensional $\forall$ cpt open K of G.

$GL_n(F)$: $n \times n$ invertible matrices with entries in F

$GL_n(\mathcal{O})$

Recall

$GL_n(\mathcal{O})$: the open maximal cpt subgp of $G(F)$.

$1 + \varpi M_n(\mathcal{O})$: cpt & open

gives a fundamental system of open nbd of 1 in $GL(F)$.

Def 3.5 (π, V): irr. adm. rep'n of $G(F)$ is spherical (or unramified) w.r.t. $G(\mathcal{O})$ if

Def 3.6 Hecke algebra of G w.r.t. $G(\mathcal{O})$

$$\mathcal{H}(G(F), G(\mathcal{O})) = C_c^\infty(G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}))$$

i.e. an algebra of $G(\mathcal{O})$-bi-invariant (smooth) fns of cpt support.

Fact 3.7 ① $\mathcal{H}(G(F), G(\mathcal{O}))$ $\circlearrowright$ $V^{G(\mathcal{O})}$

②

$\Rightarrow$ If π : unramified, then $\dim V_\pi^{G(\mathcal{O})} = 1$

$\Rightarrow$ define the character χ_π of $\mathcal{H}(G(F), G(\mathcal{O}))$ as follows

$$\pi(h) \cdot v = \chi_\pi(h) v \qquad h \in \mathcal{H}(G(F), G(\mathcal{O}))$$

Thm 3.8 Satake isomorphism

$$\mathcal{H}(G(F), G(\mathcal{O})) \cong \mathbb{C}[\hat{G}_{ss}]^{\hat{G}}.$$

$\hat{G}$-orbits

Thm 3.9 (Unramified LLC)

$$\left\{\begin{array}{l}\text{isomorphism classes of}\\ \text{unramified}\\ \text{rep'ns}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l}\text{characters of}\\ \text{Hecke algebra}\end{array}\right\}$$

$\dim V^{G(\mathcal{O})} = 1$

$\cong$ Satake isomorphism

$$\left\{\begin{array}{l}\hat{G}\text{-conjugacy classes}\\ \text{of semisimple elts of } \hat{G}\end{array}\right\}$$

"GL_n case: $\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$"

Ex 3.10 π: an irr. adm. unramified rep'n of $GL_2(F)$

(Fact: $\exists$ unramified characters χ_1, χ_2 of F^* s.t. π: unique unramified quotient of $\mathrm{Ind}(\chi_1 \otimes \chi_2)$)

Then $\pi \overset{\text{LLC}}{\sim} \begin{pmatrix} \chi_1(\varpi) & \\ & \chi_2(\varpi) \end{pmatrix} \in \hat{GL}_2$

"Satake parameter" of π

(Note 3.11) Why important?

Fact Tensor product theorem (ring of adele

$\pi = \otimes' \pi_v$ automorphic rep'n of $G(\mathbb{A})$

"global object"

$\otimes'$: restricted tensor product relative to the K_v-fixed vector for almost all v.

π_v is unramified for almost all v (2)

Restricted tensor product of vector spaces

$\{V_v\}_{v \le \infty}$ S: a finite set of primes

For $v \notin S$, let ξ_v° be a given distinguished vector in V_v.

The restricted tensor product of $\{V_v\}_{v\le\infty}$ w.r.t. $\{\xi_v^\circ\}_{v\notin S}$ is

$$\text{span}\left\{ \xi = \otimes'_{v\le\infty} \xi_v \;\middle|\; \xi_v \in V_v \ \forall v,\ \xi_v = \xi_v^\circ \text{ for all but finitely many } v \right\}$$

Def 3.12 L-ftns (unramified case)

π: irr. adm. unramified rep'n of M

$\xleftrightarrow{\text{LLC}}$

A_π: semisimple conjugacy class in $\hat{M}$

r: f-d complex repn of $\hat{M}$. Then

We define the local Langlands L-ftns as follows

$$L(s, \pi, r) := \underline{\qquad\qquad}, \quad s \in \mathbb{C}$$

Ex 3.13 π_i unramified of $GL_2(F)$ $i=1,2$

$\pi_1 \sim A_{\pi_1} = \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} \in \hat{GL}_2 = GL_2(\mathbb{C})$

$\pi_2 \sim A_{\pi_2} = \begin{pmatrix} \beta_1 & \\ & \beta_2 \end{pmatrix}$ //

The standard L-ftn of π_1

$$L(s, \pi_1) = \underline{\qquad\qquad}$$

The Rankin-Selberg L-fction of π_1 and π_2

$$L(s, \pi_1 \times \pi_2) = \det\left(I - q^{-s} A_{\pi_1} \otimes A_{\pi_2}\right)^{-1}$$

$$=$$

Q. How do we define $L(s, \pi, r)$ if π is ramified?

Summary

$\pi = \bigotimes_v \pi_v$ automorphic cuspidal repn of G

$r: \hat{G} \to GL_N(\mathbb{C})$ finite-dimensional repn

Since π_v is unramified for almost all v,

For $v \notin S$, $\pi_v \sim \{A_{\pi_v}\} \subset \hat{G}$ $(\Leftrightarrow v \notin S)$ (S: finite set)

& $L(s, \pi_v, r) = \det\left(I - r(A_{\pi_v}) q_v^{-s}\right)^{-1}$.

Conj 3.14 Langlands

$\forall v \in S$, we can define

L-ftn $L(s, \pi_v, r)$ (of the form $P(q_v^{-s})^{-1}$ ← polynomial)

ε-factors (" $A q_v^{-ms}$)

s.t. (P2) and (P3) holds for $L(s, \pi, r) = \prod_v L(s, \pi_v, r)$

(P2) Functional Equation $\tilde{r}(g) = {}^t r(g)^{-1}$

$$L(s, \pi, r) = (\varepsilon\text{-factors}) L(1-s, \pi, \tilde{r})$$

(P3) has a meromorphic continuation to all of $\mathbb{C}$.

Method: <

A. The Langlands-Shahidi method does this for certain r in the case π is generic repn.

L4: generic $\int$ def ? important ?

Many L-ftns appear in the constant terms of Eisenstein series associated to cuspidal repns of Levi subgp $\subsetneq$

E.S. $\Rightarrow$ L-ftns

L 4

Langlands - Shahidi method I

(generic rep'ns)

local coefficients

⇓

γ-factors

⇓

L-fns

Main tools

Eisenstein series

Intertwining operator

4. Whittaker functionals and the Jacquet functor

Goal: ① Definition of generic rep'ns ②

② Property of " "

(⇔ def & Property of Whittaker functionals)

G = conn. red. gp/F or GL_n

F: non-Archimedean local field of char 0.

ψ: a non-trivial additive char of F (complex valued)

$U(F)$: upper triangular unipotent matrices in $GL_n(F)$

χ: a character of $U(F)$ defined by

$$\chi(u) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right), \quad u = (u_{i,j}) \in U(F)$$

(π, V): a smooth irred. rep'n of $GL_n(F)$.

Def 4.1 (χ-generic Whittaker functional for π)

$\lambda : V \to \mathbb{C}$: a linear functional s.t.

$$\lambda(\pi(u)x) = \chi(u)\lambda(x)$$

for all $u \in U(F)$, $x \in V$.

Thm 4.2 (uniqueness of Whittaker models)
(π, V): an irred adm. rep'n of $GL_n(F)$
Then

$$\dim\left(\begin{array}{c}\text{the space of Whittaker functionals} \\ \text{on } V\end{array}\right) \leq 1.$$

We fix a Whittaker functional λ.

Def 4.3 (Whittaker model of (π, V))
W_v: function on $GL_n(F)$ of the form

$$W_v(g) = \lambda(\pi(g)v) \quad \text{for } v \in V$$

$g \in GL_n(F)$; λ: Whittaker functional; $\pi(g)v \in V$

Def 4.4 An ir. adm. rep'n π of $G(F)$ is called χ-generic if $\exists$ non-zero χ-generic Whittaker functional for G.

Thm 4.5

(π, V) : an ir. adm. rep'n of $G(F)$

$$\dim \left(\begin{array}{c} \text{the space of} \\ \chi\text{-generic Whittaker functional for } \pi \end{array} \right) \leq 1$$

b. Jacquet functor

(π, V): a smooth rep'n of $GL_2(F)$

$B(F) = T(F)N(F)$: Borel subgroup, $P = MN$, M normalizes N

$T(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in GL_2(F) \right\}$ torus of diagonal elts in $GL_2(F)$

$N(F) = U(F) = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right\}$

$V_N := \operatorname{span}\{\pi(n)v - v \mid n \in N(F), v \in V\} \leq V$ subspace

$T(F) \curvearrowright V_N$

$\pi(t)(\pi(n)v - v)$

$= \pi(\underbrace{tnt^{-1}}_{n'})\,\pi(t)v - \pi(t)v$

$= \pi(n')v' - v' \in V_N$ ($v' = \pi(t)v$)

$\Rightarrow J(V) = V/V_N \curvearrowleft T(F)$ the largest quotient of V on which N acts trivially (the space of co invariant)

Def 4.6 $\pi : T(F) \longrightarrow \operatorname{End}(J(V))$

$(\pi_N, J(V))$: corresponding rep'n of $T(F)$

↑ Jacquet module

obtained by restricting to $B(F)$ & taking the quotient $T(F) \cong B(F)/N(F)$

Def 4.7 Jacquet functor $(J = r_M^G = r_T^{GL_2})$

$r_T^{GL_2}$ Rep'n of $GL_2(F) \longrightarrow$ Rep'n of $T(F)$

$(\pi, V) \longmapsto (\pi_N, J(V))$

Similarly for G: conn. red. gp /F

$P = MN$ parabolic subgp of G

M: Levi, N: unipotent radical of P

we define Jacquet functor

Thm 4.8 Jacquet functor is exact.

$$0 \to V' \hookrightarrow V \twoheadrightarrow V'' \to 0$$

short exact sequence of $B(F)$-module

$$\Rightarrow 0 \to V'_N \to V_N \to V''_N \to 0 \text{ is exact}$$

Relation with Whittaker functional

(twisted Jacquet functor)

Similarly as V_N, we define

ψ: fixed nontrivial additive character of F

$$V_{N,\psi} = \text{span}\{\pi(n)v - \psi_N(n)v \mid n \in N(F), v \in V\}$$

$$J_\psi(V) = V/V_{N,\psi}$$

Fact 4.9

① The functor J_ψ is exact.

② λ is W.F. (Whittaker functional)

$\Leftrightarrow \lambda(\pi(n)v) = \psi(n)\lambda(v)$

$\Leftrightarrow \lambda$ annihilates $V_{N,\psi}$

$\Leftrightarrow \lambda$ factor through $J_\psi(V)$.

③ (π, V): an irred adm. rep'ns of $GL_2(F)$.

Then $\dim J_\psi(V) \leq 1$.

C. Langlands-Shahidi

C.0 Preliminaries (Induced rep'n, intertwining operator

Def 4.10 Induced rep'n 연습

G: GL_n . GL_2 GL_n

M Levi sbgp $\begin{pmatrix} a & \\ & b \end{pmatrix}$ $\begin{pmatrix} GL_{n_1} & \\ & GL_{n_2} \end{pmatrix}$

$$\mathrm{Ind}_P^G : \mathrm{Rep}(M) \longrightarrow \mathrm{Rep}(G)$$

($P = MN$)

(σ, V_σ) : irred adm. rep'n of $M(F)$

$\pi = \mathrm{Ind}(\sigma)$

$$V_\pi = \left\{ f : G \to V_\sigma \;\middle|\; f(mng) = \delta_P^{1/2}(m)\sigma(m)f(g) \right\}$$

($\delta_P^{1/2}$: modular character of P)

$m \in M, n \in N, g \in G$

& f is locally constant

GL_2 case : $\delta_P\begin{pmatrix} a & \\ & b \end{pmatrix} = |a||b|^{-1}$

$\exists$ an open sbgp K_0 of G s.t.

$f(gk) = f(g)$ $\forall g \in G, k \in K_0$.

If G is a locally cpt gp,

$\exists$ left-invariant and right-invariant measures on G called left and right Haar measures, respectively.

If they are the same, then G is called unimodular.

Ex) $GL(n, F)$, unipotent, cpt, abelian gp

non-unimodular gp : parabolic subgps.

연습 (upper triangular matrix의 modular character 계산)

- $GL(n, F)$

$$b = \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & 1 & \ddots & x_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_n \end{pmatrix}$$

(1) $d_R b = \prod_{1 \le i < j \le n} dx_{ij} \prod_{i=1}^{n} d^\times y_i$ is a right Haar measure on $B(F)$

(2) $d_L b = |y_1|^{1-n} |y_2|^{3-n} \cdots |y_n|^{n-1} d_R b$ is a left Haar measure on $B(F)$

Def Since $d_R b = \delta_B(b)\, d_L b$,

$$\delta_B(b) = |y_1|^{n-1} |y_2|^{n-3} \cdots |y_n|^{1-n}.$$

Note $d^\times y_i$ is both left and right Haar measure

$\downarrow$ $\frac{1}{|y_i|} dy_i$

Why important?

Thm 4.11 (Jacquet)

π: irr. adm. rep'n of $G(F)$

$\exists\ P = MN$ parabolic sbgp of G

& σ: supercuspidal rep'n of $M(F)$ s.t.

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) \neq 0$$

i.e. π is a subrep'n of $\mathrm{Ind}_P^G(\sigma)$.

Note 4.12 This is NOT a full classification of adm. rep'n

- Ind( ) might be reducible
- Ind( ) might have more than one irred sub.

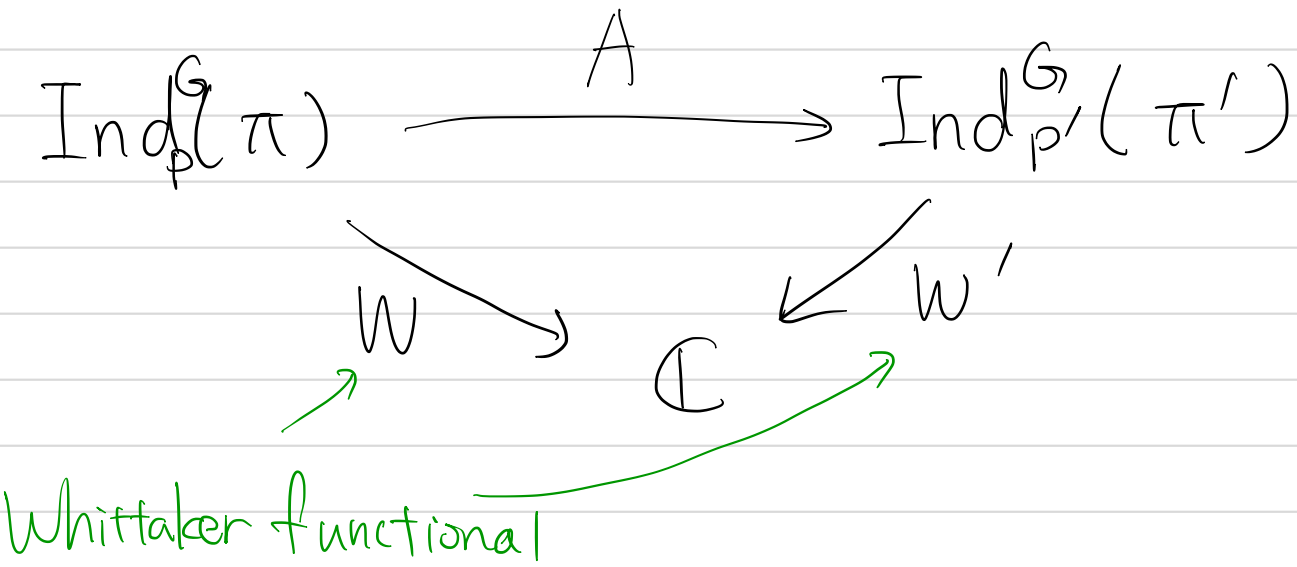
Subject: Classification of discrete series.

Def 4.13 Intertwining operator

$(\pi_1, V_1) \xrightarrow{T: \text{a linear map}} (\pi_2, V_2)$ s.t.

$$T \circ \pi_1(g) = \pi_2(g) \circ T \quad \forall g \in G$$

Def 4.14 Local coefficient



C.1 local coefficient (maximal case)

G : split

maximal torus which is split

(B, T) Borel pair & Δ : the set of simple roots.

Thm 4.15

$\exists$ one-to-one

- parabolic sbgp containing B $\longleftrightarrow$ subset of Δ

$$P_\theta \longleftarrow \sim\sim\sim \cup\ \theta \quad (\cup\ \Delta)$$

- P_θ is called maximal if $\exists \alpha \in \Delta$ s.t. $\theta = \Delta \setminus \{\alpha\}$

From now on, we consider only maximal case.

$W(T) = N_G(T)/T$ Weyl gp

$\exists w_0 \in W(T)$ s.t. $w_0(\theta) \subset \Delta$ & $w_0(\alpha) < 0$

Ex 4.1b GL_2 case $w_0 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ act like

$$w_0 \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} w_0^{-1} = \begin{pmatrix} t_2 & \\ & t_1 \end{pmatrix}$$

Setting (maximal case, i.e. $\Theta = \Delta \setminus \alpha_0$)

σ : irred. adm. generic rep'n of $M(F)$

$$I(s,\sigma) := \mathrm{Ind}_P^G\left(\sigma \otimes q^{\langle s\tilde{\alpha}, H_M(\cdot)\rangle}\right)$$

where ρ denotes the sum of positive roots in N

& $\tilde{\alpha} = \langle \rho, \alpha^\vee \rangle^{-1} \rho$

- $\mathbb{Z}$-pairing between characters of (G,T) and cocharacters
- $\alpha^\vee$: the coroot of the root α

$s\tilde{\alpha} \in$ complexification of $X^*(M)$.

$$q^{\langle \chi, H_M(m)\rangle} = |\chi(m)|_F \quad \forall m \in M,\ \chi \in X^*(M)_F$$

Rep'n space $V(s,\sigma)$: v.s. of smooth ftn

$f : G \longrightarrow V_\sigma$ s.t.

$\delta_P(a^*b) = \left|\frac{a}{b}\right|$

$\delta_P = 2\rho_P$

$$f(mng) = \sigma(m)\, q^{\langle s\tilde{\alpha} + \rho_P, H_P(m)\rangle} f(g)$$

$$\big(I(s,\sigma)(g) f\big)(h) = f(hg)$$

"right action"

Intertwining operator

$$A(s, \pi, w_0) f(g) = \int_{N'(F) = w_0 N w_0^{-1}} f(w_0^{-1} n g)\, dn$$

$$I(s, \pi) \longrightarrow I(w_0(s), w_0(\pi))$$

Check if $f \in I(s, \pi)$, then $A(s, \pi, w_0) f \in I(w_0(s), w_0(\pi))$

Def 4.17 Local coefficient (maximal case)

$$I(s\tilde{\alpha}, \sigma) \longrightarrow I(w(s\tilde{\alpha}), w(\sigma))$$

$$A(s\tilde{\alpha}, \sigma, w): I(s, \sigma) \longrightarrow I(w(s), w(\sigma))$$

$$\| \text{ def}$$

$$\mathrm{Ind}_P^G\left(\pi \otimes q^{\langle s\tilde{\alpha}, H_M(\cdot)\rangle}\right)$$

$\lambda_\chi(s\tilde{\alpha}, \sigma)$ $\qquad$ $\lambda_\chi(w(s\tilde{\alpha}), w(\sigma))$

$\mathbb{C}$

$$C_\chi(s\tilde{\alpha}, \sigma, w)\ \lambda_\chi(w(s\tilde{\alpha}), w(\sigma))\ A(s\tilde{\alpha}, \sigma, w) = \lambda_\chi(s\tilde{\alpha}, \sigma)$$

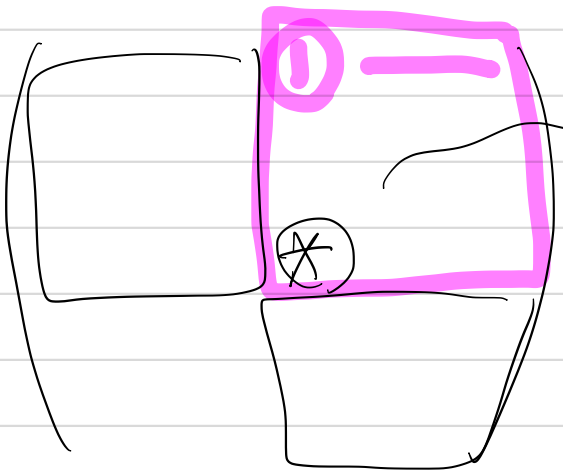
$C_\chi(s\tilde{\alpha}, \sigma, w)$: local coefficient $\in \mathbb{C}$

"rational function of q^{-s}"

Ex 4.18 (Shahidi, 1984) case of $GL_m \times GL_n$

$G = GL_{m+n}$ $\quad M = GL_m \times GL_n$

π_m (resp. π_n) irr. adm. generic repn of $GL_m(F)$ (resp. $GL_n(F)$)



unique simple root α in N

i.e. $\Theta = \Delta \setminus \{\alpha\}$ & $M = M_\Theta$

$P_\Theta = M_\Theta N_\Theta$

ρ_P: sum of positive roots in N

$$\begin{array}{lll} x_1 - x_{m+1}, & x_1 - x_{m+2}, \cdots, & x_1 - x_{m+n} \\ x_2 - x_{m+1}, & \cdots & x_2 - x_{m+n} \\ x_m - x_{m+1}, & \vdots \cdots & x_m - x_{m+n} \end{array}$$

$$+ \quad n(x_1 + \cdots + x_m) - m(x_{m+1} + \cdots + x_{m+n})$$

$\alpha = x_m - x_{m+1}$

$$\tilde{\alpha} = \frac{1}{\langle \rho_P, \hat{\alpha} \rangle} \rho_P \qquad \langle \rho_P, \hat{\alpha} \rangle = m+n$$

$(n, n, \cdots, n, -m, -m, \cdots, -m) \cdot (0 \cdots 0 1 -1)$

$\exists\, w_0 \in W$ s.t. $w_0(\Theta) \subset \Delta$ & $w_0(\alpha) < 0$

$$C_X\left(\overbrace{(m+n)^{-1}s\rho_p}^{s\tilde{\alpha}}, \pi_n \otimes \tilde{\pi}_m, w_0\right)$$

$$= \gamma(s, \pi_n \times \pi_m, \psi)$$

where R.H.S. defined by Jacquet, Piatetski-Shapiro, Shalika "Rankin-Selberg convolution"

Note4.19 LLC is known for GL_n/F

$$\pi_n \overset{\text{LLC}}{\sim} \varphi_n : W_F' \to GL_n(\mathbb{C})$$
$$\pi_m \overset{\text{LLC}}{\sim} \varphi_m : W_F' \to GL_m(\mathbb{C})$$

Then $L(s, \pi_n \times \pi_m) = L(s, \varphi_n \otimes \varphi_m)$

& $\varepsilon(s, \pi_n \times \pi_m, \psi) = \varepsilon(s, \varphi_n \otimes \varphi_m, \psi)$

where R.H.S. are Artin factor attached to

$\varphi_n \otimes \varphi_m : W_F' \to GL_{nm}(\mathbb{C})$.

"Analytic side"

Therefore local coefficient (Langlands-Shahidi)

= Artin factors

"Number theory side"

C.2 γ-factors (LS-method)

Prop 4.20 F: archimedean or G is unramified.

(Fact: LLC is known for any conn. red. gp)

ϕ : Langlands parameter of G $W_F' \to {}^L M$ (Langlands & Shahidi)

Then $C_\chi(s\tilde{\alpha}, G, w_0)$

$$= \prod_{i=1}^{m} \gamma(is, r_i \circ \phi, \psi)$$

(Here r : the adjoint action of ${}^L M$ on ${}^L N$)

$$r = \bigoplus_{i=1}^{m} r_i$$

r_i : irreducible (determined by root condition)

${}^L N_w$

$N_w = U \cap wN\bar{w}^{-1}$

self-associate case

$wN\bar{w}^{-1} = N$

(Note: this naturally appear in the calculation of intertwining operator in the unramified case)

adjoint action ↷

$\langle \tilde{\alpha}, \beta^\vee \rangle = i$

$V_i = \text{span}\{ E_{\beta^\vee} \in L_n \mid \langle \tilde{\alpha}, \beta^\vee \rangle = i \}$

Define (naturally)

$$\gamma(s, 6, r_i, \psi) = \gamma(s, r_i \circ \phi, \psi)$$

$$L(s, 6, r_i) = L(s, r_i \circ \phi)$$

$$\varepsilon(s, 6, r_i, \psi) = \varepsilon(s, r_i \circ \phi, \psi)$$

Then $\gamma(s, 6, r_i, \psi) = \varepsilon(s, 6, r_i, \psi) \dfrac{L(1-s, 6, \tilde{r}_i)}{L(s, 6, r_i)}$

Main theorem in Langlands-Shahidi

Thm 4.2.1 F: p-adic field

ρ: χ-generic supercuspidal repn of $M(F)$

Then $\exists$ unique m complex functions

$\gamma(s, \rho, r_i, \psi)$ s.t.

(1) $C_\chi(s\tilde{\alpha}, \rho, w_0) = \prod_{i=1}^{m} \gamma(is, \rho, r_i, \psi)$.

(2) Functional Equation

$\pi = \bigotimes_v \pi_v$ generic cuspidal repn

S: finite set of place

$$L_S(s, \pi, r_i) = \prod_{v \in S} \gamma(s, \pi_v, r_i, \psi_v) L_S(1-s, \tilde{\pi}, \tilde{r}_i).$$

③ Multiplicativity of γ-factors.

ex) 6: unique subrep'n of Steinberg

$$\mathrm{Ind}\left(|\,|^{\frac{p-1}{2}}\rho \otimes |\,|^{\frac{p-1}{2}-1}\rho \otimes \cdots \otimes |\,|^{-\frac{p-1}{2}}\rho\right)$$

where ρ: supercuspidal rep'n of GL_k

$$\mathrm{Ind}(6 \otimes \tilde{\rho}) \hookrightarrow \mathrm{Ind}\left(|\,|^{\frac{p-1}{2}}\rho \otimes \cdots \otimes |\,|^{-\frac{p-1}{2}}\rho \otimes \tilde{\rho}\right)$$

Then ③ implies

$$\gamma(s, 6 \otimes \tilde{\rho}, \psi) = \prod_{i=0}^{p-1} \gamma\left(s, |\,|^{\frac{p-1}{2}-i}\rho \times \tilde{\rho}, \psi\right)$$

$$I(s, 6 \otimes \tilde{\rho}) \subset \mathrm{Ind}\left(|\,|^{\frac{s}{2}+\frac{p-1}{2}}\rho \otimes |\,|^{\frac{s}{2}+\frac{p+1}{2}-1}\rho \otimes \cdots \otimes |\,|^{\frac{s}{2}-\frac{p-1}{2}} \otimes |\,|^{-\frac{s}{2}}\tilde{\rho}\right)$$

$$\gamma(s, 6 \times \tilde{\rho}, \psi) = \prod_{i=0}^{p-1} \gamma\left(s + \frac{p-1}{2} - i, \rho \times \tilde{\rho}, \psi\right)$$

Note Similar property of ③ for L-fton does NOT hold?

(due to cancellation)

Remark 4.22 If $\phi : W_F \to {}^L M$ is parametrization of ρ, we EXPECT $\gamma(s, \rho, r_i, \psi)$

$$\| \quad \gamma(s, r_i \circ \phi, \psi).$$

But few cases is unknown.

GL_n ✓ (Shahidi) Classical gps ✓ (Shahidi-Kim-Cogdell.P.-Krishnamurthy)

Thm 4.23 ((Heiermann-K, K) $GSpin$ ✓ local Langlands

general case ?

similitude groups ?

exceptional groups !

C.3 L-ftns from LS method

tempered case σ: tempered, generic

Recall $\gamma(s, \sigma, r_i, \psi)$: rational ftn of q^{-s}

$$= \frac{P_{\sigma,i}(q^{-s})}{\text{polynomial of } q^{-s}}$$

← numerator s.t. ① $P_{\sigma,i}(t)$ is poly.
② $P_{\sigma,i}(0) = 1$.

Then define

$$L(s, \sigma, r_i) := P_{\sigma,i}(q^{-s})^{-1}$$

$$L(s, \tilde{\sigma}, \tilde{r}_i) := P_{\tilde{\sigma},i}(q^{-s})^{-1}$$

$$\varepsilon(s, \sigma, r_i, \psi) := \gamma(s, \sigma, r_i, \psi) \frac{L(s, \sigma, r_i)}{L(1-s, \tilde{\sigma}, \tilde{r}_i)}$$

↑ monomial in q^{-s}

$(B q^{-ms})$

$\left(\because \text{Fact: } \gamma(s, \sigma, r_i, \psi)\gamma(1-s, \tilde{\sigma}, \tilde{r}_i, \psi) = 1\right)$

general case σ: non-tempered, generic

Follow Langlands classification

Def 4.14 σ_{temp} : tempered rep'n

$I(\nu, \sigma_{temp})$ is in the Langlands situation

if $\mathrm{Re}(\langle \nu, \alpha^\vee \rangle) > 0 \quad \forall \alpha \in \Delta - \theta$

($\nu \in \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$)

$\mathfrak{a}^* = X^*(M) \otimes_{\mathbb{Z}} \mathbb{R} = X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$

(A: connected component of the center of M)

(So far we only consider $\nu = s\tilde{\alpha}$ with maximal case)

π_1, π_2 : tempered

$\pi_1 \times \pi_2$

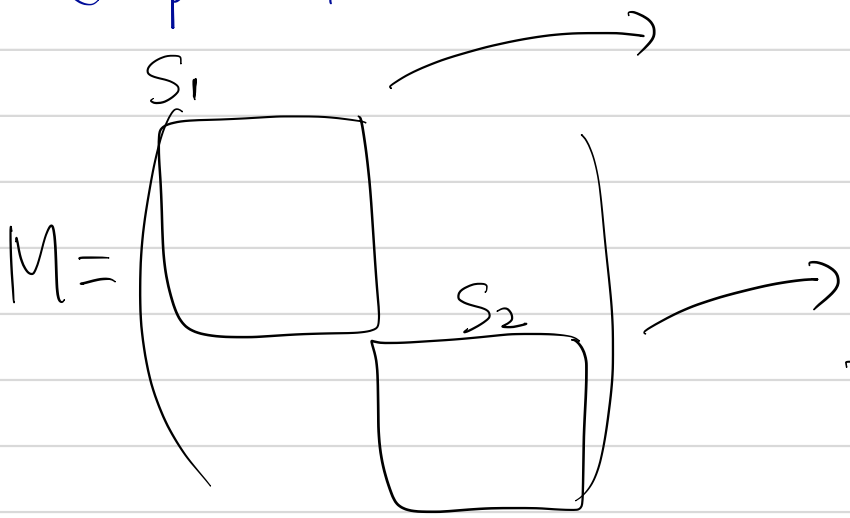
Ex 4.25 $GL_m \times GL_n \subset GL_{m+n}$

$\Delta = \{\chi_1 - \chi_2, \cdots, \underbrace{\chi_m - \chi_{m+1}}_{\alpha}, \cdots, \chi_{n-1} - \chi_n\}$

$\theta = \Delta \setminus \{\alpha\}$

$I(s\tilde{\alpha}, \pi_1 \times \pi_2) = \mathrm{Ind}(\pi_1 \times \pi_2 \otimes q^{\langle s\tilde{\alpha}, H_P(\ ) \rangle})$

"complexification of $X^*(M)$"



$\nu = s_1(\chi_1 + \cdots + \chi_m) + s_2(\chi_{m+1} + \cdots + \chi_{m+n})$

$I(s_1, s_2, \pi_1 \otimes \pi_2) = \mathrm{Ind}(\pi_1 \otimes |\ |^{s_1} \otimes \pi_2 \otimes |\ |^{s_2})$

Langlands situation? $\mathrm{Re}(\langle \nu, \alpha^\vee \rangle) > 0 \ \forall \alpha \in \Delta - \theta$

Our case $\mathrm{Re}(\langle \nu, \alpha^\vee \rangle) > 0 \quad \alpha = \chi_m - \chi_{m+1}$

$\Leftrightarrow \mathrm{Re}(s_1) - \mathrm{Re}(s_2) > 0$

Thm 4.26 (Langlands classification)

F: local field

Assume $I(\nu, \sigma)$ is in Langlands' situation.

Then $I(\nu, \sigma)$ has a unique irreducible quotient, called Langlands' quotient, denote it by $J(\nu, \sigma)$.

Any irr. adm. rep'n π of $G(F)$ occurs uniquely as some $J(\nu, \sigma)$.

(P, ν, σ) : Langlands' data for π.

adm. / temp.) Langlands classification

d.s.

s.c.

L5

Langlands - Shahidi method II

(Eisenstein series)

Summary

Intertwining operators (local coefficients)

$\Downarrow$

L-ftns from Langlands-Shahidi

Fact • It appears in the constant term of Eisenstein series

• non-constant terms gives some properties of L-ftns

E.S.	L-ftns
meromorphic continuation	$\Rightarrow$ Langlands M.C. ~ quasi-split
F.E.	Shahidi & F.E. (non-constant term)

⋮

L5. LS-method II : Eisenstein series
(Global object)

K: number field G: conn. reductive gp / K

$\mathbb{A}_K$: adeles of K

π: cuspidal rep'n of $M(\mathbb{A})$

$P = P_\theta = MN$, $\theta = \Delta - \{\alpha_b\}$

Def 5.1 Global induced rep'n

$$I(s,\pi) = \bigotimes_v I(s,\pi_v)$$

↑ restricted tensor product

$$V(s,\pi) = \left\{ f \in \bigotimes_{v\in S} V(s,\pi_v) \otimes \bigotimes_{v\notin S} f_v^\circ \;\middle|\; f_v^\circ \text{ is s.t. } f_v^\circ(k_v) = 1 \ \forall k_v \in G(\mathcal{O}_v) \right\}$$

Similarly, global intertwining operator

$$M(s,\pi)f(g) = \int_{N'(\mathbb{A})} f(w_0^{-1}ng)\,dn$$

where N' is the unipotent radical of $P' = P_{w_0(\theta)}$

Then $M(s,\pi) = \bigotimes_v A(s,\pi_v,w_0)$

$$\therefore T(s,\pi) \longrightarrow I(-s, w_0(\pi))$$

Thm 5.2 (Langlands)

π is automorphic rep'n of $G(\mathbb{A})$

iff π is a subquotient of $I(s,\sigma)$

for some cuspidal rep'n σ of $M(\mathbb{A})$

$\pi = \bigotimes_v \pi_v$

π_v is a subquotient of $I(\nu,\sigma_v)$ $\forall v$

π_v is unramified for almost all v

Def 5.3 Eisenstein series

For $f_s = \phi(\,) q^{\langle s\tilde{\alpha} + \rho_P, H_P(\,)\rangle} \in I(s,\pi)$

$$E(s,\pi,\phi,g) = \sum_{\gamma \in P(K)\backslash G(K)} \frac{f_s(\gamma g)}{\phi(\gamma g)\, q^{\langle s\tilde{\alpha}+\rho_P, H_P(\gamma g)\rangle}}.$$

Ex 5.4

ρ_P: $\frac{1}{2}$ (sum of positive roots in N)

δ_P: modulus character $= \frac{d_{\text{left}}\, p}{d_{\text{right}}\, p}$ $\qquad \delta_P(m) = |\det{}^{''} Ad(m)|$

$|2\rho_P(m)| = \delta_P(m)$

$G = GL_2/\mathbb{Q} \qquad P = B$: Borel sbgp

$$B(\mathbb{Q})\backslash G(\mathbb{Q}) \simeq SL_2(\mathbb{Z}) \cap B(\mathbb{Q}) \backslash SL_2(\mathbb{Z})$$

$f = q^{\langle s\tilde{\alpha}+\rho_P, H_P(\,)\rangle}$ & $g = \begin{pmatrix} y^{\frac{1}{2}} & y^{-\frac{1}{2}} x \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}, y>0$

(i.e. $\phi = 1$)

⋮ exercise

$$E(s,1,g) = \sum_{\gamma \in SL_2(\mathbb{Z}) \cap B(\mathbb{Q}) \backslash SL_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^{\frac{s+1}{2}}, \qquad z = x+iy$$

$$SL_2(\mathbb{Z}) \cap B(\mathbb{Q}) = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$$

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}\, z}{|cz+d|^2}$$

$$= \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{y^{\frac{s+1}{2}}}{|cz+d|^{s+1}}$$

$\exists n$ s.t.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \quad \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

$$\Leftrightarrow \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Ex 5.5

$$E(z,s) := \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{y^s}{|cz+d|^{2s}}, \quad z = x+iy \in \mathbb{C}$$

Fourier expansion: $E(x+iy,s) = \sum_{n\in\mathbb{Z}} a_n(y,s)\, e^{2\pi i n x}$

Fact $\sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{y^s}{|cz+d|^{2s}} = \sum_{m=1}^{\infty} \sum_{\substack{(c',d')\in\mathbb{Z}^2 \\ (c',d')=1}} \frac{y^s}{|mc'z+md'|^{2s}}$

$$= 2\sum_{m=1}^{\infty} \frac{1}{m^{2s}}\, E(z,s) = 2\zeta(2s)\, E(z,s)$$

Riemann-zeta

$$a_0(y,s) = \int_0^1 E(x+iy,s)\,dx$$

$$= \frac{1}{2\cdot\zeta(2s)} \int_0^1 \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{y^s}{|cz+d|^{2s}}\,dx \quad \text{by Fact}$$

$$= \frac{1}{2\zeta(2s)} \left[\int_0^1 \sum_{d\in\mathbb{Z}} \frac{y^s}{|d|^{2s}}\,dx + \int_0^1 \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ c\neq 0}} \frac{y^s}{|cz+d|^{2s}}\,dx \right]$$

$$= y^s + \frac{1}{2\zeta(2s)} \int_0^1 \sum \frac{y^s}{\left((x+d)^2+c^2y^2\right)^s}\,dx$$

$$= y^s + \frac{1}{\zeta(2s)} \sum_{c=1}^{\infty} \sum_{d \bmod c} \int_{-\infty}^{\infty} \frac{y^s}{\left((x+d)^2+c^2y^2\right)^s}\,dx$$

Substitute $x \mapsto x - \frac{d}{c}$

$$= y^s + \frac{1}{2\zeta(2s)} \sum_{c=1}^{\infty} \frac{y^s}{c^{2s}} \int_{-\infty}^{\infty} \frac{1}{(x^2+y^2)^s} dx \left(\sum_{d \bmod c}^{c} 1 \right)$$

$$= y^s + \frac{\zeta(2s-1)}{2\zeta(2s)} y^s \int_{-\infty}^{\infty} \frac{1}{(x^2+y^2)^s} ds = \frac{\pi^{-s+\frac{1}{2}} \Gamma(s-\frac{1}{2}) y^{1-s}}{(\frac{y}{\pi})^s \Gamma(s)}$$

$$= y^s + \frac{\pi^{-s+\frac{1}{2}} \Gamma(s-\frac{1}{2}) \zeta(2s-1)}{\pi^{-s} \Gamma(s) \zeta(2s)} y^{1-s} = y^s + \frac{L(2s-1)}{L(2s)} y^{1-s}.$$

$\Rightarrow$ constant term of Fourier series.

$$a_0(y,s) = y^s + \frac{L(2s-1)}{L(2s)} y^{1-s}.$$

Maass의 보조정리 $E(z,s) = (\text{상수}) E(z,1-s)$

$a_0(y,s)$와 $a_0(y,1-s)$ 비교

$\Rightarrow$ $E(z,s) = \frac{L(2s-1)}{L(2s)} E(z,1-s)$: 함수 방정식

$n=1 \Rightarrow$ $a_1(y,s) = \frac{L(2s-1)}{L(2s)} a_1(y,1-s)$

for $n \neq 0$ non-constant term of Fourier series.

$$a_n(y,s) = \frac{2|n|^{s-\frac{1}{2}} \sigma_{1-2s}(|n|) \sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y)}{L(2s)}$$

$\rightsquigarrow$ $L(s) = L(1-s)$

Basic properties of E.S.

- $E(s, \pi, f_s, g)$ and $M(s, \pi)$ have meromorphic continuation to all of $\mathbb{C}$
- $E(-s, w_0(\pi), M(s,\pi) f_s, g) = E(s, \pi, f_s, g)$

Continue to assume: π: cuspidal rep'n of $M(\mathbb{A})$, $P = P_\theta = MN$, $\theta = \Delta - \{\alpha_k\}$

Def 5.6 Constant term of $E(s, \pi, \phi, g)$ along a parabolic sbgp $Q = M_Q N_Q$.

$$E_Q(s, \pi, \phi, g) = \int_{N_Q(F)\backslash N_Q(\mathbb{A})} E(s, \pi, \phi, ng)\, dn.$$

Thm 5.7 Unless $Q = P$, $P_{w_0(\theta)}$, $E_Q(s, \pi, \phi, g) = 0$.

If P is self-conjugate, i.e. $P = P'$, then

$$E(s, \pi, \phi, g) = \phi(g)\, q^{\langle s\tilde{\alpha} + \rho_P, H_P(g)\rangle} + M(s, \pi)\phi(g)\, q^{\langle -s\tilde{\alpha} + \rho_P, H_P(g)\rangle}.$$

A1.

- The Artin reciprocity map of local class field theory

$$r : F_v^\times \longrightarrow \mathrm{Gal}(\bar{F}_v/F_v)^{ab}$$

has dense image, inducing an injection r^*

$\{$ Chars of $\mathrm{Gal}(\bar{F}_v/F_v)\} \hookrightarrow \{$ Chars of $F_v^\times\}$

If $r^*(\rho_v) = \chi_v$,

$$L(s, \rho_v) = L(s, \chi_v)$$

the local Artin L-factor

$$L(s, \chi_v) = \begin{cases} (1 - \chi_v(\bar{w}_v) q_v^{-s})^{-1} & \text{if } \chi_v \text{ is unramified} \\ 1 & \text{otherwise} \end{cases}$$

A 2

for $\Gamma(1)$

(P1) If f : normalized ($a_1(f)=1$) cuspidal Hecke eigenform,

then $L(s,f) = \prod_p \left(1 - a_p(f)p^{-s} + p^{k-1}\cdot p^{-2s}\right)^{-1}$

rough idea of pf

Fact ① $a_m(f)\,a_n(f) = a_{mn}(f)$ if $(m,n)=1$

② $a_{p^{r+1}}(f) - a_p(f)\,a_{p^r}(f) + p^{k-1}a_{p^{r-1}}(f) = 0$.

(By the theory of Hecke operators)

$$L(s,f) = \sum \frac{a_n(f)}{n^s} = \prod_p \left(1 + \frac{a_p(f)}{p^s} + \frac{a_{p^2}(f)}{p^{2s}} + \cdots\right)$$

$$= \prod_p \left(\sum_{r=0}^{\infty} a_{p^r}(f)\,p^{-rs}\right)$$

Fact ② $\Rightarrow \left(1 - a_p(f)X + p^{k-1}X^2\right)\left(\sum_{r=0}^{\infty} a_{p^r}(f)X^r\right)$

$$= \sum_{r=0}^{\infty} a_{p^r}(f)X^r - \sum_{r=0}^{\infty} a_p(f)\,a_{p^r}(f)X^{r+1} + \sum_{r=0}^{\infty} p^{k-1}a_{p^r}(f)X^{r+2}$$

$$= a_1(f) + a_p(f)X - a_p(f)\,a_1(f)X = 1$$

$$\therefore\ L(s,f) = \prod_p \frac{1}{1 - a_p(f)p^{-s} + p^{k-1}p^{-2s}}$$